PRINCIPLES OF ANALYSIS
LECTURE 2 - COLLECTIONS AND RELATIONS

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1. Collections

A *collection* is a set whose elements are themselves sets.

Let X be a set. The collection of all subsets of X is called the *power set* of X and is denoted $\mathcal{P}(X)$.

Let $\mathcal{C}$ be a collection of subsets of X; then $\mathcal{C} \subset \mathcal{P}(X)$. Define the *intersection* and *union* of the collection by

- $\cap\mathcal{C} = \{a \in A \mid a \in C \text{ for all } C \in \mathcal{C}\}$
- $\cup\mathcal{C} = \{a \in A \mid a \in C \text{ for some } C \in \mathcal{C}\}$

If $\mathcal{C}$ contains two subsets of X, this definition concurs with our previous definition for the union of two sets.

Let A and B sets. The collection of all functions from A to B is denoted $\mathcal{F}(A, B)$, and is a subset of $\mathcal{P}(A \times B)$.

2. Families

Let A and X be sets. A *family* of subsets of X indexed by A is the image of an injective function $Y : A \to \mathcal{P}(X)$. For each $a \in A$, the set $Y(a)$ may be denoted by Y_a. The family itself may denoted by $\{Y_a \subset X \mid a \in A\}$

Let $\{Y_a \subset X \mid a \in A\}$ be a family of subsets of a set X. The *intersection* and *union* of the family is defined by

- $\cap_{a \in A} Y_a = \{x \in X \mid x \in Y_a \text{ for all } a \in A\}$;
- $\cup_{a \in A} Y_a = \{x \in X \mid x \in Y_a \text{ for some } a \in A\}$;

Let X be a set and let $\mathcal{C} \subset \mathcal{P}(X)$ be a collection of subsets of X. Then $\mathcal{C}$ is a family of subsets of X, indexed by itself via the identity function. Our definitions of intersection and union of a family of subsets concur with our definitions of intersections and union of a collection of subsets under this correspondence.

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3. Cartesian Product of a Family

Let X be a set and let $\mathcal{C} = \{Y_a \subset X \mid a \in A\}$ be a family of subsets of X. Let $Y = \cup_{a \in A} Y_a$.

The *cartesian product* of $\mathcal{C}$ is denoted by $\times\mathcal{C}$ or by $\times_{a\in A} Y_a$ and is defined to be the collection of all functions from A into the union of the family such that each element of a is mapped to an element of Y_a:

$$\times_{a\in A} Y_a = \{f \in \mathcal{F}(A, Y) \mid f(a) \in Y_a\}.$$

We needed to define the cartesian product of two sets in order to define function, which in turn we have used to define the cartesian product of more than two sets. These definitions concur according to the following proposition.

Proposition 1. *Let X be a set and let $Y_1, Y_2 \subset X$. Let $A = \{1, 2\}$ and let $Y = Y_1 \cup Y_2$. For any $x_1, x_2 \in X$, define a function $f_{x_1,x_2} : A \to X$ by $f(1) = x_1$ and $f(2) = x_2$. Define a function*

$$\phi : Y_1 \times Y_2 \to \{f \in \mathcal{F}(A, Y) \mid f(a) \in Y_a\} \quad \text{by} \quad \phi(x_1, x_2) = f_{x_1,x_2}.$$

Then ϕ is a bijection.

4. Relations

A *relation* on a set A is a subset of $R \subset A \times A$. If $(a, b) \in R$, we may indicate this by writing aRb; that is $(a, b) \in R \Leftrightarrow aRb$.

A relation is called *reflexive* if aRa for every $a \in A$.

A relation is called *symmetric* if $aRb \Leftrightarrow bRa$ for every $a, b \in A$.

A relation is called *antisymmetric* if aRb and bRa implies $a = b$.

A relation is called *transitive* if aRb and bRc implies aRc.

A relation is called *definite* if either aRb or bRa for every $a, b \in A$.

A *partial order* on A is a relation which is reflexive, antisymmetric, and transitive. A *total order* on A is a definite partial order.

An *equivalence relation* on A is a relation which is reflexive, symmetric, and transitive.

If $\equiv$ is an equivalence relation on A and $a \in A$, the *equivalence class* of a is the set

$$[a]_\equiv = \{b \in A \mid a \equiv b\}.$$

A set is *nonempty* if it is not equal to the empty set. Two sets A, B are called *disjoint* if $A \cap B = \varnothing$. A collection $\mathcal{C} \subset \mathcal{P}(A)$ of subsets of A is said to be *pairwise disjoint* if every every distinct pair of members of $\mathcal{C}$ are disjoint. A collection $\mathcal{C} \subset \mathcal{P}(A)$ of subsets of A is said to *cover* A if $\cup\mathcal{C} = A$.

A *partition* of A is a collection $\mathcal{C} \subset \mathcal{P}(A)$ of subsets of A such that

- $\varnothing \notin \mathcal{C}$;
- $\cup\mathcal{C} = A$;
- $A, B \in \mathcal{C} \Rightarrow A \cap B = \varnothing$ or $A = B$.

That is, a partition of A is a pairwise disjoint collection of nonempty subsets of A which covers A.

If $\equiv$ is an equivalence relation A, then the collection of equivalence classes under $\equiv$ is a partition of A. If $\mathcal{C}$ is a partition of A, we may define an equivalence relation $\equiv$ on A by $a \equiv b$ if and only if they are in the same subset of the partition.

If $f : A \to X$ is a surjective function, then the relation $\equiv$ on A defined by $a \equiv b \Leftrightarrow f(a) = f(b)$ is an equivalence relation which partitions A into blocks of elements which are sent to the same place by f. There is a natural bijective function from the partition into X given by sending each block to the appropriate element in X.

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